

# Torsional displacements and stresses in non-homogeneous soil

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A study is presented of the axially symmetric deformation of a non-homogeneous elastic halfspace subjected to torsional shear tractions that are linearly distributed over a circular portion of the otherwise free surface. The problem is formulated in terms of Hankel integral transforms and an inverse procedure is implemented in which the determination of the type of non-homogeneity constitutes part of the problem. A numerical parametric study shows the effect of the degree of soil inhomogeneity on the distribution of circumferential displacements and shear stresses in the medium. With increasing non-homogeneity stresses affect soil at greater vertical and lesser radial distances. Gibson's law of the Winkler-type behaviour of linearly inhomogeneous, incompressible soils can only qualitatively apply in this case, as the surface displacements tend to become proportional to the applied local pressure and decay rapidly away from the torsionally loaded area in strongly non-homogeneous deposits.

Il est présenté une étude de la déformation axi symétrique d'un semi-espace élastique non homogène soumis à des efforts de traction par torsion répartis linéairement sur une zone circulaire de la surface libre par ailleurs. Le problème est formulé en fonction des transformées intégrales de Henkel et un procédé inverse est appliqué au sein duquel la détermination du type de non homogénéité constitue une partie du problème. Une étude exhaustive des paramètres numériques montre l'effet du degré de non homogénéité du sol sur la répartition des déplacements périphériques et des efforts de cisaillement dans le substrat. Les contraintes affectent le terrain sur des distances verticales plus grandes et des distances radiales moins grandes en fonction de l'accroissement de la non homogénéité. La loi de Gibson sur le comportement de type Winkler des sols linéairement non homogènes et incompressibles n'est applicable que qualitativement dans ce cas, car les déplacements superficiels ont tendance à devenir proportionnels à la pression appliquée localement et à s'altérer en s'éloignant rapidement de la zone sous charge de torsion, dans des dépôts de terre fortement non homogènes.

## INTRODUCTION

Frequently foundations transmit torsional loads on to the soil. This happens whenever asymmetric horizontal forces act on the superstructure, as may be the case, for instance, during wind storms and earthquake shaking. Torsional loading of the

supporting soil is caused by the horizontal movements of the antennae of radar towers, the rotation of unbalanced masses of reciprocating engines and by pneumatic tyres during cornering. Even bridge piers subjected to asymmetric stream-water forces may produce torsional loads on their foundation and nuclear power plant structures, for example, must be designed against tornado and impact forces that would introduce severe torsional shearing of the supporting soil.

Following Reissner (1937) and Reissner & Sagoci (1944) others (e.g. Sneddon, 1951; Bycroft, 1955; Collins, 1962; Kaldjian, 1971; Waas, 1972; Novak & Sachs, 1973; Veletsos & Nair, 1974; Luco, 1976; Kausel & Ushijima, 1979) have investigated particular aspects of the torsional loading problem. Most have studied the dynamic moment-rotation relation of harmonically excited

## NOTATION

$b$	parameter defining increase of $G$ with depth
$B(\xi)$	integration constant, function of Hankel transform variable
$c$	rate of inhomogeneity $bR$
${}_2F_1(g, h; q; x)$	Gauss hypergeometric function with parameters $g, h$ and $q$
$G$	shear modulus
$G_0$	surface value of shear modulus
$I_a(c; \rho, \zeta)$	dimensionless influence function for circumferential displacement
$I_{z\theta}(c; \rho, \zeta)$	dimensionless influence function for shear stress components
$I_{r\theta}(c; \rho, \zeta)$	first kind Bessel function of order $a$ and argument $g$
$J_a(g)$	
$p_0$	magnitude of imposed torsional shear stress at $r = R$
$R$	radius of loaded area
$r, \theta, z$	polar cylindrical co-ordinates
$v$	circumferential displacement
$V$	first order Hankel transform of $v$ (equation (4))
$\zeta$	normalized depth $z/R$
$\xi$	Hankel transform parameter
$\rho$	normalized radial distance $r/R$
$\tau_{z\theta}, \tau_{r\theta}$	non-vanishing shear stress components (Fig. 1)
$\psi, u, f$	functions in inverse procedure (equation (7))

Discussion on this Paper closes 1 March 1982. For further details see inside back cover.

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rigid circular surface or embedded foundations, using analytical, numerical and approximate techniques. The static rotation has also been obtained from such studies as the limiting case of zero frequency oscillation. However, Gerrard & Harrison (1974) have given complete solutions for the torsional stress, displacement and strain distributions within a homogeneous soil mass having cross-anisotropic or isotropic properties. Linear torsional shear stress and linear torsional shear displacement types of loading were considered, the latter simulating the conditions imposed by a rigid footing perfectly bonded to the soil surface.

These studies all make the assumption of soil homogeneity, yet it is well known that soil stiffness varies continuously with depth below the surface in a manner which depends on the geologic and loading history of the particular deposit. For instance, even within a uniform layer, soil stiffness increases with effective overburden pressure and degree of overconsolidation, which are both continuous functions of depth. Experimental evidence suggests that the stiffness of London clay increases approximately linearly with depth (e.g. Ward, Samuels & Butler, 1959; Wroth, 1971).

Considerable attention has been given to the problem of determining stresses and deformations caused in non-homogeneous soil deposits by vertical, horizontal and moment loading under both static (Lekhnitskii, 1962; Gibson, 1967, 1974; Awojobi & Gibson, 1973; Awojobi, 1974; Chuaprasert & Kassir, 1974; Gibson & Kalsi, 1974) and dynamic (Awojobi, 1972; Gazetas, 1980) conditions. Gibson (1974) presents a list of publications on the subject. The most striking finding of these studies—Gibson's law—is that an incompressible soil, with modulus which increases linearly with depth (being zero at the surface), responds to vertical loads as a Winkler medium, i.e. as a uniform bed of independent springs, with the surface settlement being proportional to the local pressure. For other types of loading and forms of soil inhomogeneity, this law is not strictly valid; nevertheless, in most cases the displacements of the free surface away from the loaded area decay faster than the elastic homogeneous theories predict, and the distribution of soil reactions against rigid foundations may be uniform, in accordance with a Winkler-type rather than a homogeneous continuum hypothesis.

To the Author's knowledge, no similar study has been undertaken with regard to torsional loading. This Paper investigates the effect of inhomogeneity on the axially symmetric elastic deformation arising in a soil deposit which is subjected to torsional shear tractions distributed linearly over a circular portion of the surface. Soil inhomogeneity

is described by a shear modulus monotonically increasing with depth; it is evident that such a variation in modulus can adequately represent a broad class of soil deposits, ranging from mildly ( $c < 0.2$ ) to strongly ( $c > 0.5$ ) inhomogeneous.<sup>1</sup>

The problem is formulated in terms of Hankel integral transforms. To obtain analytical expressions for displacement and stresses, in transform space, an inverse procedure has been devised in which the type of inhomogeneity has to be determined. As analytical inversion of the Hankel transforms of the resulting expression appears intractable, a simple numerical integration scheme is used to obtain the complete solution for stress and displacement distributions in the soil.

Parametric studies show that soil inhomogeneity has a considerable effect on all stress and displacement components in the medium. As the degree of inhomogeneity increases, torsional shear stresses affect the soil at greater vertical and lesser radial distances; surface circumferential displacements decay even more rapidly away from the loading area. This extends, although only in qualitative terms, the applicability of Gibson's law to torsionally loaded non-homogeneous soils of the type considered in the Paper.

#### THE PROBLEM AND GOVERNING EQUATIONS

Of interest are the stresses and displacements arising in a non-homogeneous isotropic halfspace ( $z \geq 0$ ) when axisymmetric torsional shear tractions are imposed on a circular portion ( $r \leq R$ ) of the surface ( $z = 0$ ) while the remaining portion ( $r > R$ ) of the surface is stress-free. The centre O of the loaded area is taken to be the origin of cylindrical polar co-ordinates  $r, \theta, z$ , as shown in Fig. 1, and linear variation of the imposed tractions with  $r$  is considered.

Due to the symmetry of the problem, there is no dependence on  $\theta$  and the components of the displacement in the  $r$  and  $z$  directions are zero at all points, e.g. Reissner & Sagoci (1944), Collins (1962). Consequently, only circumferential displacements  $u(r, z)$  occur, yielding two non-zero components of stress (Fig. 1(b))

$$\tau_{z\theta}(r, z) = G(z) \partial v / \partial z \quad (1a)$$

$$\tau_{r\theta}(r, z) = G(z) \left( \frac{\partial v}{\partial r} - \frac{v}{r} \right) \quad (1b)$$

<sup>1</sup> Moreover, as only shear deformations are produced by an axisymmetric torsional loading of an elastic medium, soil compressibility has no effect on the distribution of stresses and displacements studied in this Paper. The conclusions drawn are therefore applicable to compressible and incompressible soils (i.e. under drained and undrained conditions).

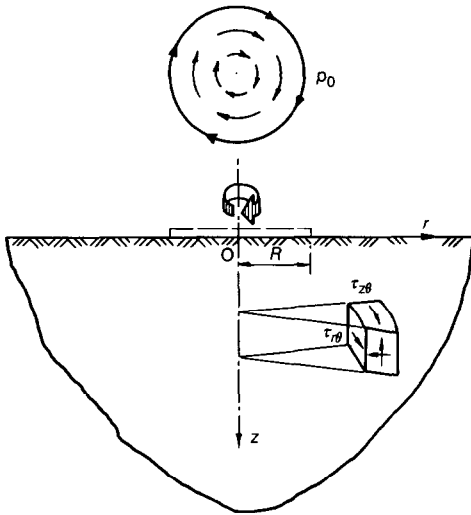


Fig. 1. Loading geometry, co-ordinate axes and non-vanishing shear stress components

in which  $G(z)$  is the yet unspecified shear modulus at a depth  $z$ . In the absence of body forces, equilibrium in the circumferential direction yields

$$G(z) \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} \right) + \frac{\partial}{\partial z} \left[ G(z) \frac{\partial v}{\partial z} \right] = 0 \quad (2)$$

The appropriate boundary conditions at the surface are

$$\tau_{z\theta}(r, 0) = p_0 r/R, \quad r \leq R \quad (3a)$$

$$\tau_{z\theta}(r, 0) = 0, \quad r > R \quad (3b)$$

in which  $R$  is the radius of the loaded area and the constant  $p_0$  represents the magnitude of the imposed torsional shear stress at  $r = R$  (Fig. 1). Far from the surface it is required that

$$v(r, z) = \tau_{r\theta}(r, z) = \tau_{z\theta}(r, z) = 0 \quad \sqrt{(r^2 + z^2)} = \infty \quad (3c)$$

In view of the axial symmetry, the problem can be expeditiously formulated by recourse to the theory of Hankel transforms (Sneddon, 1951). The first order Hankel transform of  $v(r, z)$  is introduced, i.e.

$$V(\xi, z) = H_1 \{ v(r, z); \xi \} = \int_0^\infty r v J_1(\xi r) dr \quad (4a)$$

where  $v(r, z)$  can be recovered from  $V(\xi, z)$  using the inversion theorem

$$v(r, z) = H_1^{-1} \{ V(\xi, z); r \} = \int_0^\infty \xi V J_1(\xi r) d\xi \quad (4b)$$

$J_1(x)$  denotes the first kind and first order Bessel function of  $x$ .

Both sides of equation (2) are then multiplied by  $r J_1(\xi r)$  and integrated with respect to  $r$  from 0 to  $\infty$ . As (Sneddon, 1951)

$$H_1 \left\{ \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} \right); \xi \right\} = -\xi^2 V \quad (5a)$$

$$H_1 \left\{ \frac{\partial^n v}{\partial z^n}; \xi \right\} = \frac{d^n V}{dz^n} \quad (5b)$$

this leads to

$$G(z) \xi^2 V = \frac{d}{dz} \left[ G(z) \frac{dV}{dz} \right] \quad (6)$$

## INVERSE PROCEDURE AND SOLUTION IN TRANSFORMED SPACE

It is expedient not only to obtain an analytical solution of equation (6), but also to have the solution in a form that will allow the direct analytical evaluation of the constants of integration. This is not possible with an arbitrary variation law of the shear modulus with depth  $G(z)$ . An inverse procedure is developed here in which the function describing the variation of modulus with depth is not prespecified, but its determination constitutes part of the problem. To this end, a system of differential equations is constructed that transforms equation (6) into an equation of the same form with the corresponding equation for a torsionally loaded homogeneous halfspace. The solutions of the system and the latter equation are then combined to obtain the solution for a torsionally loaded halfspace, and simultaneously the functional form of the shear modulus  $G(z)$  is determined. A similar inverse procedure has been developed by Schreyer (1977), who studied the one-dimensional wave propagation into an inhomogeneous halfspace, and by Gazetas (1981), who investigated the shear vibrations of vertically inhomogeneous earth dams.

Consider a transformation of the dependent variable  $V(\xi, z)$  and the independent variable  $z$  in equation (6) of the form

$$V(\xi, z) = \psi(z) u(\xi, z_0) \quad (7a)$$

$$z_0 = f(z) \quad (7b)$$

with the restrictions

$$\psi(z) > 0 \quad \text{and} \quad f(z) > 0 \quad \text{for} \quad z > 0 \quad (8a)$$

$$f(0) = 0 \quad (8b)$$

This transformation is used to reduce the governing equation (6) to

$$\xi^2 u(\xi, z_0) = a^2 \frac{d^2 u(\xi, z_0)}{dz_0^2} \quad (9)$$

in which  $a$  is a positive constant. The general solution of equation (9) is

$$u(\xi, z_0) = B(\xi) \exp(-\xi z_0/a) \quad (10)$$

in view of the boundary condition (3c). The

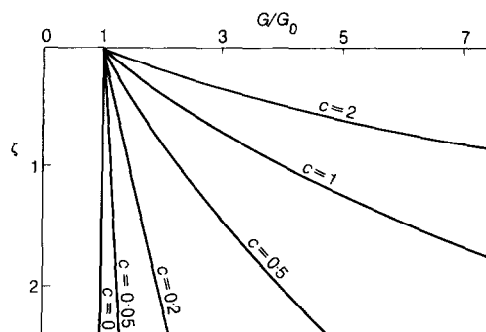


Fig. 2. Variation of shear modulus of soil with depth below the surface, for several values of the rate of non-homogeneity  $c$

integration constant  $B(\xi)$  then has to be determined from the boundary conditions of the problem ((3a) and (3b)).

Expressions (7a) and (7b) are now substituted into equation (6), with  $F_{,x}$  and  $F_{,xx}$  used to denote the first and second derivatives of  $F$  with respect to  $x$ . This gives

$$G\xi^2\psi u = (Gf_{,z}^2\psi)u_{,z0z0} + (2G\psi_{,z}f_{,z} + \psi G_{,z}f_{,z} + G\psi f_{,zz})u_{,z0} + (G_{,z}\psi_{,z} + G\psi_{,zz})u \quad (11)$$

To match equations (11) and (9),  $f$ ,  $\psi$  and  $G$  must satisfy the differential equations

$$f_{,z}^2 = a^2 \quad (12a)$$

$$2G\psi_{,z}f_{,z} + \psi G_{,z}f_{,z} + G\psi f_{,zz} = 0 \quad (12b)$$

$$G_{,z}\psi_{,z} + G\psi_{,zz} = 0 \quad (12c)$$

which, for a non-trivial solution obeying the restrictions (8a) and (8b), can be recast in the form

$$f = az \quad (13a)$$

$$(G\psi^2 f_{,z})_{,z} = 0 \quad (13b)$$

$$(G\psi_{,z})_{,z} = 0 \quad (13c)$$

Equations (13b) and (13c) simplify to

$$G\psi^2 = \text{constant} \quad (14)$$

$$G\psi_{,z} = \text{constant} \quad (15)$$

from which it directly follows by integration that

$$G = G_0(1+bz)^2 \quad (16)$$

$$\psi = 1/(1+bz) \quad (17)$$

where  $b$  and  $G_0$  are integration constants.

Combining equations (7), (17), (13a) and (10) gives the solution of equation (6)

$$V(\xi, z) = \frac{B(\xi)}{1+bz} \exp(-\xi z) \quad (18)$$

provided that the variation of shear modulus with

depth is given by equation (16).<sup>2</sup> Using a dimensionless parameter  $c = bR$ , called the rate of inhomogeneity, where  $R$  is the radius of the loaded area, equation (16) is written as

$$G/G_0 = (1+c\xi)^2 \quad (16a)$$

where  $\xi = z/R$ .

Figure 2 shows  $G/G_0$  as a function of  $\xi$  for several positive values of  $c$ . It is evident that a broad class of (inhomogeneous) soil deposits can be represented by equation (16a). For  $c = 0$ , in particular, equation (16a) reduces to  $G = G_0$  and represents the homogeneous halfspace. For small values of  $c$ , say less than 2.0, and small depths, say less than  $2R$ , equation (16a) can be reasonably accurately approximated by a straight line and, thus, may represent the so-called generalized Gibson soil (Gibson, 1967, 1974; Awojobi, 1972, 1974; Awojobi & Gibson, 1973). With larger values of  $c$ , equation (16a) models more extreme cases of soil inhomogeneity.

The expedience of extending the vertical axis in Fig. 2 only to a value of  $\xi$  of about 2.5 may be questioned. However, below a depth of  $2R$  the stresses in the soil are negligibly small—less than 5% of the maximum applied torsional shear stress—and the exact variation of  $G$  below this depth hardly influences deformations and stresses in the soil.

The transformed function  $V(\xi, z)$  and equation (4b) are then used to recover the displacement  $v(r, z)$ . Similarly, equations (1a) and (1b) are transformed and the inversion theorem (Sneddon, 1951) is used to give

$$\tau_{z\theta} = G(z) \int_0^\infty \xi \frac{dV(\xi, z)}{dz} J_1(\xi r) d\xi \quad (19a)$$

$$\tau_{r\theta} = G(z) \int_0^\infty \xi^2 V(\xi, z) J_2(\xi r) d\xi \quad (19b)$$

$B(\xi)$  is then evaluated by recourse to the boundary conditions.

#### ENFORCEMENT OF BOUNDARY CONDITIONS AND SOLUTION

By substituting equation (18) into equation (19a), equations (3) can be written as

$$\int_0^\infty (\xi + b) B(\xi) \xi J_1(\xi r) d\xi = \begin{cases} -\frac{p_0}{G_0} \frac{r}{R}, & 0 \leq r \leq R \\ 0, & r > R \end{cases} \quad (20a)$$

$$(20b)$$

<sup>2</sup> Direct substitution shows that equation (18) is a solution of equation (6) if  $G$  varies according to equation (16).

By introducing the substitutions

$$\left. \begin{aligned} x &= \xi R \\ \rho &= r/R \\ M &= B(x/R^3)(x+c) \end{aligned} \right\} \quad (21)$$

equations (20a) and (20b) are reduced to

$$\int_0^\infty M(x) J_1(\rho x) dx = \begin{cases} -\frac{p_0}{G_0} \rho, & 0 \leq \rho \leq 1 \\ 0, & \rho > 1 \end{cases} \quad (22a)$$

$$(22b)$$

These equations belong to the class of dual integral equations solved by Busbridge (1938). The function  $M(x)$  is given by (Sneddon, 1951)

$$M(x) = -x J_1(x) \int_0^1 \frac{p_0}{G_0} y^3 dy - \int_0^1 w^2 dw \int_0^1 \frac{p_0}{G_0} y w(x y)^2 J_2(x y) dy \quad (23a)$$

Performing the integrations and making use of the recurrence relations of the Bessel functions (Watson, 1944), simplifies expression (23a) to

$$M(x) = -\frac{p_0}{G_0} J_2(x) \quad (23b)$$

The integration constant  $B(\xi)$  is obtained from equations (21) and thus  $V(\xi, z)$  is fully determined from equation (18). Equations (4b) and (19) give for the displacement and stresses

$$v(r, z) = -\frac{p_0 R}{G_0} I_v(c; \rho, \zeta) \quad (24)$$

$$\tau_{z\theta}(r, z) = p_0 I_{z\theta}(c; \rho, \zeta) \quad (25)$$

$$\tau_{r\theta}(r, z) = -p_0 I_{r\theta}(c; \rho, \zeta) \quad (26)$$

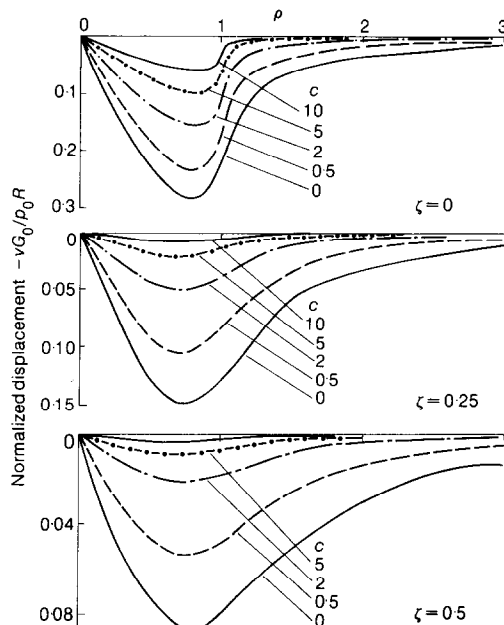


Fig. 3. Radial distribution of circumferential displacements on three horizontal planes with  $\zeta = 0, 0.25$  and  $0.5$

in which

$$I_v = (1 + c\zeta)^{-1} \times \int_0^\infty \frac{J_2(x) J_1(\rho x)}{x + c} \exp(-\zeta x) dx \quad (24a)$$

$$I_{z\theta} = \int_0^\infty \frac{J_2(x) J_1(\rho x)}{x + c} \times \exp(-\zeta x) [x(1 + c\zeta) + c] dx \quad (25a)$$

$$I_{r\theta} = (1 + c\zeta) \int_0^\infty \frac{x \exp(-\zeta x)}{x + c} \times J_2(x) J_2(\rho x) dx \quad (26a)$$

Table 1. Effect of upper limit and number of intervals used to approximate the infinite integrals of the influence function

$\zeta$	$\rho$	Numerical values of $I_v$ for $c = 0.0001^*$				$I_v$ for $c = 0$ from Gerrard & Harrison (1974)
		$A = 100, N = 400$	$A = 100, N = 200$	$A = 50, N = 200$	$A = 25, N = 100$	
0.00	0.50	0.22507	0.22507	0.22498	0.22371	0.2252
	1.00	0.21216	0.21216	0.21208	0.21176	0.2122
	1.50	0.06344	0.06344	0.06338	0.06421	0.0635
	2.00	0.03354	0.03354	0.03349	0.03377	0.0335
0.25	0.50	0.12712	0.12712	0.12706	0.12712	0.1282
	1.00	0.12665	0.12665	0.12658	0.12664	0.1267
	1.50	0.05913	0.05913	0.05905	0.05912	0.0586
	2.00	0.03246	0.03246	0.03238	0.03245	0.0325

\*  $A$  = Upper limit.  $N$  = Number of intervals of the integration scheme.

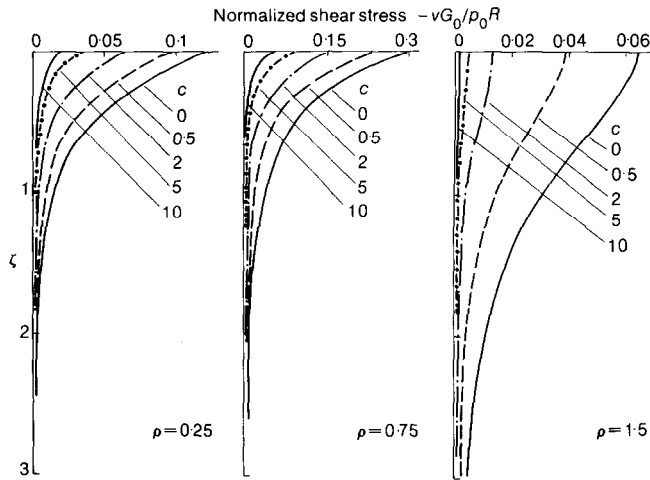


Fig. 4. Vertical distribution of circumferential displacements on three radial planes with  $\rho = 0.25, 0.75$  and  $1.5$

It appears that analytical evaluation of these integrals is not possible. However, their numerical evaluation presents no special difficulties. The integrands and their first derivatives possess no singularities as  $x$  varies from zero to infinity and use of Simpson's one third rule gives good accuracy when the upper limit of integration  $A$  is taken as 100 and the range of integration is divided into  $N = 200$  intervals. Table 1 shows the influence of  $A$  and  $N$  on the computed values of  $I_v$  for depths  $z = 0$  and  $z = 0.25R$ , and  $c = 0.0001$ . As a check, these values are compared with the numerical results of Gerrard & Harrison (1970) for a homogeneous halfspace, i.e. with  $c = 0$ . Even with

$A = 25$  and  $N = 100$  reasonable accuracy is achieved with the numerical scheme.

For  $c = 0$  the integral expressions for  $I_v$ ,  $I_{z\theta}$  and  $I_{r\theta}$  reduce to the theoretical expressions given by Gerrard & Harrison (1970) for a homogeneous halfspace, e.g. when  $z = 0$

$$I_v = \int_0^\infty \frac{J_1(\rho x) J_2(x)}{x} dx \quad (24b)$$

which belongs to the class of the Weber-Schafheitlin integrals (Watson, 1944) and can be evaluated analytically

$$I_v = \begin{cases} \frac{\rho}{2} {}_2F_1(1.5, -0.5; 2; \rho^2), & \rho < 1 \\ \frac{2}{3\pi}, & \rho = 1 \\ \frac{1}{8}\rho^{-2} {}_2F_1(1.5, 0.5; 3; \rho^{-2}), & \rho > 1 \end{cases} \quad (24c)$$

in which  ${}_2F_1$  denotes the Gauss hypergeometric function (Watson, 1944). Equation (24c) coincides with the expressions of Gerrard & Harrison (1974). Similarly  $I_{z\theta}$  reduces to  $\rho$  for  $\rho < 1$ , and vanishes for  $\rho > 1$  in apparent agreement with the imposed boundary conditions (3).  $I_{r\theta}$  reduces to

$$I_{r\theta} = \begin{cases} \frac{3}{8}\rho^2 {}_2F_1(2.5, 0.5; 3; \rho^2), & \rho < 1 \\ \infty, & \rho = 1 \\ \frac{3}{8\rho^3} {}_2F_1(2.5, 0.5; 3; \rho^{-2}), & \rho > 1 \end{cases} \quad (26b)$$

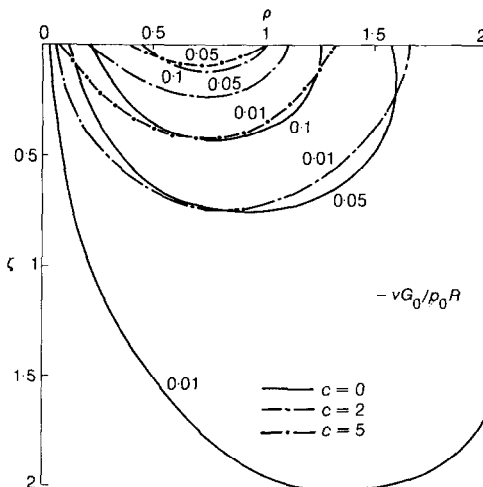


Fig. 5. Iso-displacement contours with  $-vG_0/p_0R = 0.1, 0.05$  and  $0.01$

again in agreement with Gerrard & Harrison (1974).

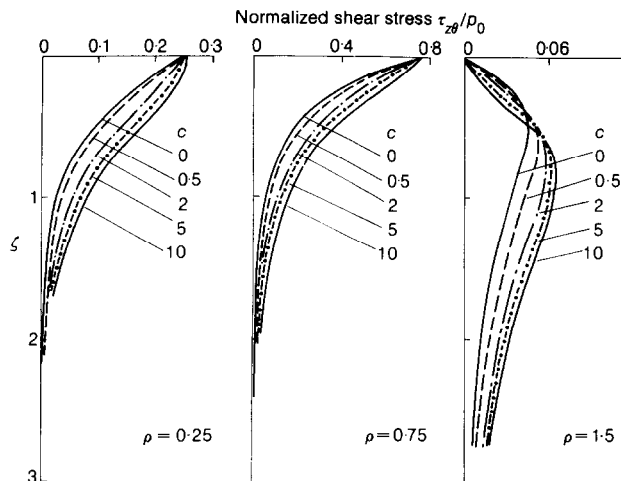


Fig. 6. Vertical distribution of shear stresses  $\tau_{z\theta}$  on three radial planes with  $\rho = 0.25, 0.75$  and  $1.5$

#### CIRCUMFERENTIAL DISPLACEMENT

The radial distribution of circumferential displacements on three horizontal planes with depths  $z = 0$  (surface),  $z = 0.25R$  and  $z = 0.5R$  is shown in Fig. 3, for several values of the rate of inhomogeneity ranging from  $c = 0$  (homogeneous medium) to  $c = 10$  (extreme case of inhomogeneity). The magnitude of displacements decreases everywhere with increasing  $c$ , as anticipated because of the increasing stiffness of the medium (Fig. 2). Moreover, surface displacements decrease much faster away from the loaded area (i.e. for  $r > R$ ) than they do under the load ( $r \leq R$ ). For example, although at  $r = 0.8R$ ,  $v_{(c=2)}/v_{(c=0)} \approx 0.158/0.286 \approx 0.552$ , at  $r = 1.2R$  the corresponding ratio reduces to  $0.031/0.108 \approx 0.287$ , i.e. to about half the preceding value. Thus, in strongly inhomogeneous deposits ( $c > 0.5$ ) surface displacements are insignificant beyond, say,  $1.5$  radii from the centre of the foundation. In addition, as  $c$  increases, the displacement profile of the loaded area tends to assume a nearly triangular shape; the surface displacements therefore become almost proportional to the applied local shear stress (equation (3)). In a Winkler medium the displacements would be exactly proportional to applied stress. Thus, Gibson's law (Awojobi, 1972, 1974; Gibson, 1974; Gibson & Kalsi, 1974) applies qualitatively to torsionally loaded, strongly non-

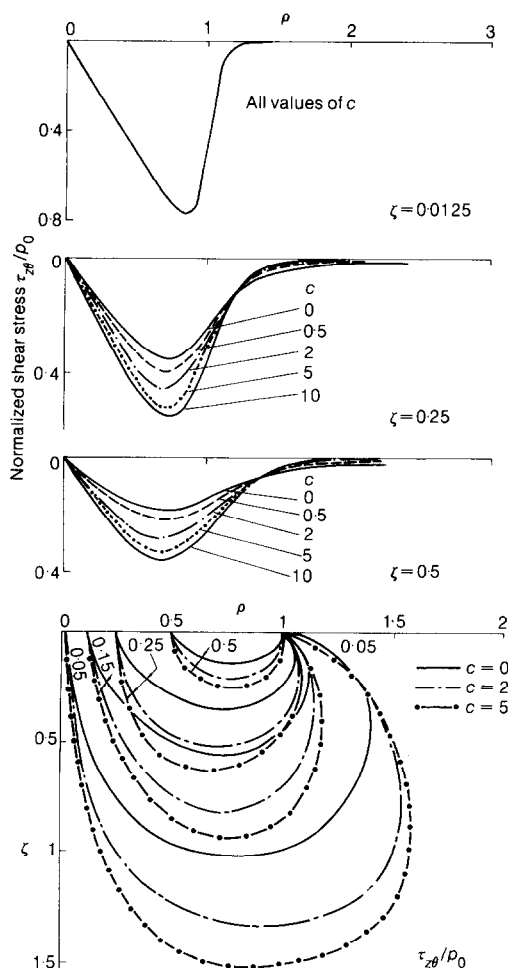


Fig. 7 (above right). Radial distribution of shear stresses  $\tau_{z\theta}$  on three horizontal planes with  $\zeta = 0.0125, 0.25$  and  $0.5$

Fig. 8 (below right). Iso-shear stress  $\tau_{z\theta}$  contours with  $\tau_{z\theta}/p_0 = 0.5, 0.25, 0.15$  and  $0.05$

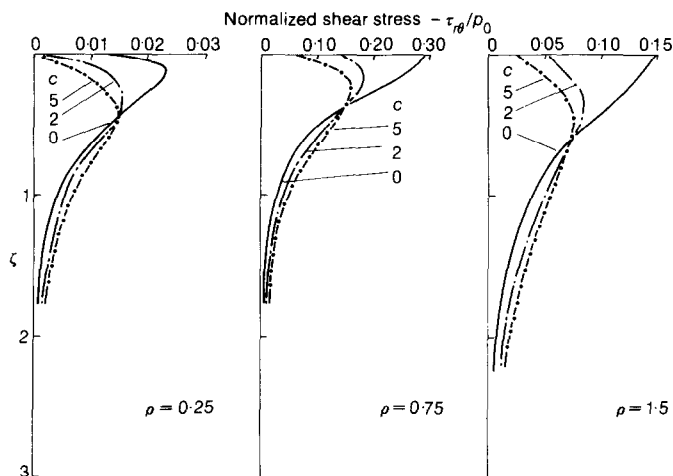


Fig. 9. Vertical distribution of shear stresses  $\tau_{r\theta}$  on three radial planes with  $\rho = 0.25, 0.75$  and  $1.5$

homogeneous soils the stiffness of which increases with depth according to equation (16).<sup>3</sup>

The vertical distribution of circumferential displacements, shown in Fig. 4 for  $c = 0-10$  and  $0.25R, 0.75R$  and  $1.5R$ , shows that negligible displacement occurs below a depth of about  $1.5R$ . Therefore, the existence of a stiffer stratum (e.g. rigid bedrock) beyond this depth would hardly influence the results presented here for a halfspace. Kausel & Ushijima (1979), using a finite element formulation, arrived at essentially the same conclusion for a rigid circular footing resting on a uniform soil stratum on rock: the torsional stiffness of the footing is independent of the stratum thickness  $H$  as long as  $H \geq 2R$ .

Figure 5 summarizes information on the distribution of circumferential displacement in the soil by showing the equal displacement contours corresponding to  $-vG_0/(p_0 R)$  values of  $0.1, 0.05$  and  $0.01$ . (For clarity, the  $0.1$  contour is not shown for  $c = 5$ .)

#### SHEAR STRESS ON HORIZONTAL PLANES

Figures 6-8 show the distribution of the shear stress  $\tau_{z\theta}$  in an inhomogeneous soil mass with  $c = 0-10$ . Three vertical and three horizontal cross-sections of the distribution are shown in Figs 6 and 7, respectively; equal stress contours are shown in Fig. 8. With increasing degree of non-

homogeneity, shear stresses affect the soil at greater vertical and lesser radial distances (near the surface), in agreement with intuition which expects stiffer material to attract larger stresses. Chuaprasert & Kassir (1974) deduced a similar conclusion with a vertically loaded inhomogeneous halfspace.

Even in strongly inhomogeneous soils (i.e.  $c > 0.5$ ), the shear stress  $\tau_{z\theta}$  attains values less than about 10% of the corresponding applied surface traction, at depths below  $1.5R$ . This value may be compared with the depth of approximately 4-5 radii required for a decrease of the normal vertical stress to a value of 10% of the applied normal surface pressure (Chuaprasert & Kassir, 1974; Poulos & Davis, 1974). The difference is partly due to differences in the distribution of the applied stresses and partly due to the fact that, on any horizontal plane, small torsional stresses at large radii contribute much more to equilibrating an applied normal force.

#### SHEAR STRESS ON RADIAL PLANES

Figures 9 and 10 show the distribution of  $\tau_{r\theta}$  in the form of three vertical and three radial profiles, respectively, and Fig. 11 shows a number of iso-stress contours. The shear stress  $\tau_{r\theta}$  bulb becomes deeper but narrower as soil inhomogeneity increases.

Large stresses  $\tau_{r\theta}$  develop only under the edge of the loading area. At the surface, in particular,  $\tau_{r\theta}$  is infinite at  $r = R$ , regardless of the value of  $c$ . A similar occurrence of infinite stresses is common with contact problems in continuum mechanics, whenever a sudden change in applied stresses or displacements occurs, e.g. under the edges of rigid footings. It is generally hypothesized that the

<sup>3</sup> Gibson's law applies exactly only to an incompressible halfspace the modulus of which increases linearly with depth, being zero at the surface. The soil studied here has a modulus increasing with depth according to equation (16) (see Fig. 2); soil compressibility has no effect on the results which, consequently, apply to compressible and incompressible soils.



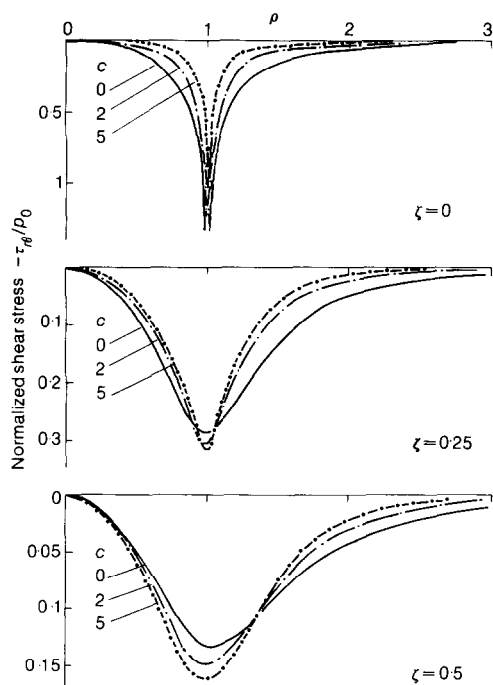


Fig. 10. Radial distribution of shear stress  $\tau_{r\theta}$  on three horizontal planes with  $\zeta = 0, 0.25$  and  $0.5$

redistribution of stresses due to yielding of the soil in a narrow region surrounding the singularity does not significantly affect the overall results.

## CONCLUSIONS

Torsional loads applied over a circular area of radius  $R$  on the surface of a deposit are only felt by the near-surface soil, up to a depth of about  $2R$ . This is true despite the fact that, in soil deposits with shear modulus increasing rapidly with depth, circumferential shear stresses influence soil at greater vertical and lesser horizontal distances than those corresponding to homogeneous deposits. Moreover, with strong non-homogeneity, circumferential surface displacements are appreciable only up to a radial distance of about  $1.5R$ , whereas within the loaded area their distribution approaches the triangular distribution of the applied torsional stresses. Thus, Gibson's law of the Winkler-spring behaviour of linearly inhomogeneous and incompressible normally loaded soils is qualitatively extended to torsionally loaded strongly inhomogeneous soils of the type considered in this Paper.

## ACKNOWLEDGEMENT

The partial financial support of this work by the US National Science Foundation is gratefully acknowledged.

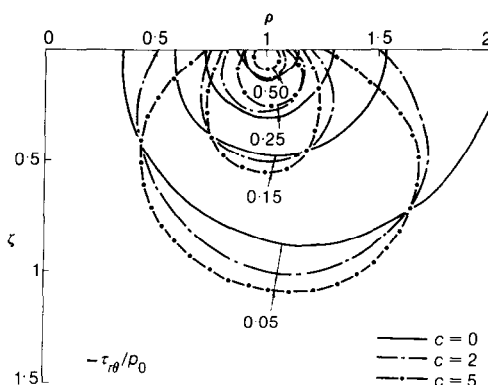


Fig. 11. Iso-shear stress  $\tau_{r\theta}$  contours with  $-\tau_{r\theta}/p_0 = 0.5, 0.25, 0.15$  and  $0.05$

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